A "milder" version of Calderón's inverse problem for anisotropic conductivities and partial data

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Abstract

Given a general symmetric elliptic operator

$$L_a := \sum_{k,j=1}^{d} \partial_k(a_{kj}\partial_j) + \sum_{k=1}^{d} a_k\partial_k - \partial_k(\overline{a_k}.) + a_0$$

we define the associated Dirichlet-to-Neumann (D-t-N) map with partial data, i.e., data supported in a part of the boundary. We prove positivity, L^p-estimates and domination properties for the semigroup associated with this D-t-N operator. Given L_a and L_b of the previous type with bounded measurable coefficients $a = \{a_{kj},\ a_k, a_0\}$ and $b = \{b_{kj},\ b_k, b_0\}$, we prove that if their partial D-t-N operators (with a_0 and b_0 replaced by $a_0 - \lambda$ and $b_0 - \lambda$) coincide for all λ, then the operators L_a and L_b, endowed with Dirichlet, mixed or Robin boundary conditions are unitarily equivalent. In the case of the Dirichlet boundary conditions, this result was proved recently by Behrndt and Rohleder [6]. We provide a different proof, based on spectral theory, which works for other boundary conditions.

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1 Introduction

Let Ω be a bounded Lipschitz domain of $\mathbb{R}^d$ with boundary $\partial\Omega$. Let Γ_0 be a closed subset of $\partial\Omega$ with $\Gamma_0 \neq \partial\Omega$ and Γ_1 its complement in $\partial\Omega$. We consider the symmetric elliptic operator on $L^2(\Omega)$ given by the formal expression:

$$L_a(\lambda) := \sum_{k,j=1}^{d} \partial_k(a_{kj}\partial_j) + \sum_{k=1}^{d} a_k\partial_k - \partial_k(\overline{a_k}.) + a_0 - \lambda$$

where $a_{kj} = \overline{a_{jk}}, a_k, a_0 = \overline{a_0} \in L^\infty(\Omega)$ and λ is a constant. We define the associated Dirichlet-to-Neumann (D-t-N) operator, $\mathcal{N}_{\Gamma_1,a}(\lambda)$, with partial data as follows:

for $\varphi \in H^{\frac{1}{2}}(\partial\Omega)$ with $\varphi = 0$ on Γ_0, one solves the Dirichlet problem

$$L_a(\lambda)u = 0 \text{ weakly in } W^{1,2}(\Omega) \text{ with } u = \varphi \text{ on } \partial\Omega, \tag{1.1}$$

and defines (in the weak sense)

$$\mathcal{N}_{\Gamma_1,a}(\lambda)\varphi := \sum_{j=1}^{d}\left(\sum_{k=1}^{d} a_{kj}\partial_k u + \overline{a_j}\varphi\right)\nu_j \text{ on } \Gamma_1. \tag{1.2}$$

Here $\nu = (\nu_1, \cdots, \nu_d)$ is the outer unit normal to the boundary of Ω. The operator $\mathcal{N}_{\Gamma_1,a}(\lambda)$ is interpreted as the conormal derivative on the boundary. It is an operator acting on $L^2(\partial\Omega)$. See Section 2 for more details.

Let us consider first the case where $a_{kj} = \sigma(x)\delta^{kj}$, $a_k = 0, k = 0, 1 \dots d$, where $\sigma \in L^\infty(\Omega)$ is bounded from below (by a positive constant). A. Calderón's well known inverse problem asks whether one could determine solely the conductivity $\sigma(x)$ from boundary measurements, i.e., from $\mathcal{N}_{\Gamma_1}(0)$. For the global boundary measurements, i.e., $\Gamma_1 = \partial\Omega$, the first global uniqueness result was proved by Sylvester and Uhlmann [27] for a C^2-smooth conductivity when $d \geq 3$. This results was extended to $C^{1+\epsilon}$-smooth conductivity by Greenleaf, Lassas and Uhlmann [12] and then by Haberman and Tataru [13] to C^1 and Lipschitz conductivity close to the identity. Haberman [14] proved the uniqueness for Lipschitz conductivity when $d = 3, 4$ and this was extended to all $d \geq 3$ by Caro and Rogers [7]. In the two-dimension case with C^2-smooth conductivity, the global uniqueness was proved by Nachman [21]. This regularity assumption was completely removed by Astala and Päivärinta [4] dealing with $\sigma \in L^\infty(\Omega)$. The inverse problem with partial data consists in proving uniqueness (either for the isotropic conductivity or for the potential) when the measurement is made only on a part of the boundary. This means that the trace of the solution u in (1.1) is supported on a set Γ_D and the D-t-N operator is known on Γ_N for some parts Γ_D and Γ_N of the boundary. This problem has been studied and there are some geometric conditions on Γ_D and Γ_N under which uniqueness is proved. We refer to Isakov [15], Kenig, Sjöstrand and Uhlmann [18], Dos Santos et al. [10], Imanuvilov, Uhlmann and Yamamoto [16] and the review paper [19] by Kenig and Salo for more references and recent developments.

Now we move to the anisotropic case. This corresponds to the general case where the conductivity is given by a general matrix a_{kj}. As pointed out by Lee and Uhlmann in [20], it is not difficult to see that a change of variables given by a diffeomorphism of Ω which is the identity on $\partial\Omega$ leads to different coefficients b_{kj} without changing the D-t-N operator on the boundary. Therefore the single coefficients a_{kj} are not uniquely determined in general. In [20], Lee and Uhlmann proved that for real-analytic coefficients the uniqueness up to a diffeomorphism holds when the dimension d is ≥ 3. The same result was proved by Astala, Lassas and Päivärinta [5] for the case $d = 2$ and L^∞-coefficients.

In [6], Behrndt and Rohleder considered general elliptic expressions L_a and L_b as above and prove that if the corresponding D-t-N operators $\mathcal{N}_{\Gamma_1,a}(\lambda)$ and $\mathcal{N}_{\Gamma_1,b}(\lambda)$ coincide for all λ in a set having an accumulation point in $\rho(L_a^D) \cap \rho(L_b^D)$ then the operators L_a^D and L_b^D are unitarily equivalent. Here L_a^D is the elliptic operator L_a with Dirichlet boundary conditions. This can be seen as a milder version of the uniqueness problem discussed above. The proof is based on the theory of extensions of symmetric operators and unique continuation results. It is assumed in [6] that the coefficients are Lipschitz continuous on $\overline{\Omega}$. We give a different proof of this result which also works for other boundary conditions. Our main result is the following.

Theorem 1.1. *Suppose that Ω is a bounded Lipchitz domain of $\mathbb{R}^d$ with $d \geq 2$. Let Γ_0 be a closed subset of $\partial\Omega$, $\Gamma_0 \neq \partial\Omega$ and Γ_1 its complement. Let $a = \{a_{kj}, a_k, a_0\}$ and $b = \{b_{kj}, b_k, b_0\}$ be bounded functions on Ω such that a_{kj} and b_{kj} satisfy the usual ellipticity condition. If $d \geq 3$ we assume in addition that the coefficients a_{kj}, b_{kj}, a_k and b_k are Lipschitz continuous on $\overline{\Omega}$.*

Suppose that $\mathcal{N}_{\Gamma_1,a}(\lambda) = \mathcal{N}_{\Gamma_1,b}(\lambda)$ for all λ in a set having an accumulation point in $\rho(L_a^D) \cap \rho(L_b^D)$. Then:
i) The operators L_a and L_b endowed with Robin boundary conditions are unitarily equivalent.
ii) The operators L_a and L_b endowed with mixed boundary conditions (Dirichlet on Γ_0 and Neumann type on Γ_1) are unitarily equivalent.
iii) The operators L_a and L_b endowed with Dirichlet boundary conditions are unitarily equivalent.

In addition, for Robin or mixed boundary conditions, the eigenfunctions associated to the same eigenvalue $\lambda \notin \sigma(L_a^D) = \sigma(L_b^D)$ coincide on the boundary of Ω.

Note that unlike [6] we do not assume regularity of the coefficients when $d = 2$.

We shall restate this theorem in a more precise way after introducing some necessary material and notation. The proof is given in Section 4. It is based on spectral theory and differs from the one in [6]. Our strategy is to use a relationship between eigenvalues of the D-t-N operator $\mathcal{N}_{\Gamma_1,a}(\lambda)$ and eigenvalues of the elliptic operator with Robin boundary conditions L_a^μ on Ω where μ is a parameter. One of the main ingredients in the proof is that each eigenvalue of the latter operator is a strictly decreasing map with respect to the parameter μ. Next, the equality of $\mathcal{N}_{\Gamma_1,a}(\lambda)$ and $\mathcal{N}_{\Gamma_1,b}(\lambda)$ allows us to prove that the spectra of L_a^μ and L_b^μ are the same and the eigenvalues have the same multiplicity. The similarity of the two elliptic operators with Dirichlet boundary conditions is obtained from the similarity of L_a^μ and L_b^μ by letting the parameter μ tend to $-\infty$. During the proof we use some ideas from the papers of Arendt and Mazzeo [2] and [3] which deal with a different subject, namely the Friendlander inequality for the eigenvalues of the Dirichlet and Neumann Laplacian on a Lipschitz domain. The ideas which we borrow from [2] and [3] are then adapted and extended to our general case of D-t-N operators with variable coefficients and partial data.

In Section 2 we define the D-t-N operator with partial data using the method of sesquilinear forms. In particular, for symmetric coefficients it is a self-adjoint operator on $L^2(\Gamma_1)$. It can be seen as an operator on $L^2(\partial\Omega)$ with a non-dense domain and which we extend by 0 to $L^2(\Gamma_0)$. Therefore one can associate with this D-t-N operator a semigroup $(T_t^{\Gamma_1})_{t\geq 0}$ acting on $L^2(\partial\Omega)$. In Section 3 we prove positivity, sub-Markovian and domination properties for such semigroups. In particular, $(T_t^{\Gamma_1})_{t\geq 0}$ extends to a contraction semigroup on $L^p(\partial\Omega)$ for all $p \in [1, \infty)$. Hence, for $\varphi_0 \in L^p(\Gamma_1)$, one obtains existence and uniqueness of the solution in $L^p(\partial\Omega)$ to the evolution problem

$$\partial_t \varphi + \mathcal{N}_{\Gamma_1,a}(\lambda)\varphi = 0, \quad \varphi(0) = \varphi_0.$$

The results of Section 3 are of independent interest and are not used in the proof of the theorem stated above.

2 The partial D-t-N operator

Let Ω be a bounded open set of $\mathbb{R}^d$ with Lipschitz boundary $\partial\Omega$. The boundary is endowed with the $(d-1)$-dimensional Hausdorff measure $d\sigma$. Let

$$a_{kj}, a_k, \tilde{a_k}, a_0 : \Omega \to \mathbb{C}$$

be bounded measurable for $1 \le k, j \le d$ and such that there exists a constant $\eta > 0$ for which

$$\operatorname{Re} \sum_{k,j=1}^{d} a_{kj}(x)\xi_k \overline{\xi_j} \ge \eta |\xi|^2 \tag{2.1}$$

for all $\xi = (\xi_1, \cdots, \xi_d) \in \mathbb{C}^d$ and a.e. $x \in \Omega$.
Let Γ_0 be an closed subset of $\partial\Omega$ and Γ_1 its complement in $\partial\Omega$.

Elliptic operators on Ω.

We consider the space

$$V = \{u \in W^{1,2}(\Omega),\ \operatorname{Tr}(u) = 0 \text{ on } \Gamma_0 = 0\}, \tag{2.2}$$

where Tr denotes the trace operator. We define the sesquilinear form

$$\mathfrak{a} : V \times V \to \mathbb{C}$$

by the expression

$$\mathfrak{a}(u,v) = \sum_{k,j=1}^{d} \int_\Omega a_{kj} \partial_k u \overline{\partial_j v}\ dx + \sum_{k=1}^{d} \int_\Omega a_k \partial_k u \overline{v} + \tilde{a_k} u \overline{\partial_k v}\ dx + a_0 u \overline{v}\ dx \tag{2.3}$$

for all $u, v \in V$. Here we use the notation ∂_j for the partial derivative $\frac{\partial}{\partial x_j}$.

It follows easily from the ellipticity assumption (2.1) that the form $\mathfrak{a}$ is quasi-accretive, i.e., there exists a constant w such that

$$\operatorname{Re} \mathfrak{a}(u,u) + w\|u\|_2^2 \ge 0 \quad \forall u \in V.$$

In addition, since V is a closed subspace of $W^{1,2}(\Omega)$ the form $\mathfrak{a}$ is closed. Therefore there exists an operator L_a associated with $\mathfrak{a}$. It is defined by

$$\begin{aligned} D(L_a) &= \{u \in V, \exists v \in L^2(\Omega) : \mathfrak{a}(u,\phi) = \int_\Omega v\overline{\phi}\ dx \ \ \forall \phi \in V\}, \\ L_a u &:= v. \end{aligned}$$

Formally, L_a is given by the expression

$$L_a u = -\sum_{k,j=1}^{d} \partial_k(a_{kj}\partial_j u) + \sum_{k=1}^{d} a_k \partial_k u - \partial_k(\tilde{a_k} u) + a_0 u. \tag{2.4}$$

In addition, following [2] or [3] we define the conormal derivative $\frac{\partial}{\partial \nu}$ in the weak sense (i.e. in $H^{-1/2}(\partial\Omega)$ the dual space of $H^{1/2}(\partial\Omega) = \operatorname{Tr}(W^{1,2}(\Omega))$), then L_a is subject to the boundary conditions

$$\begin{cases} \operatorname{Tr}(u) = 0 & \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases} \tag{2.5}$$

The conormal derivative in our case is usually interpreted as

$$\sum_{j=1}^{d}\left(\sum_{k=1}^{d} a_{kj}\partial_k u + \tilde{a_j} u\right)\nu_j,$$

where $\nu = (\nu_1, \cdots, \nu_d)$ is the outer unit normal to the boundary of Ω. For all this see [23], Chapter 4.
The condition (2.5) is a mixed boundary condition which consists in taking Dirichlet on Γ_0 and Neumann type boundary condition on Γ_1. For this reason we denote this operator by L_a^M. The subscript a refers to the fact that the coefficients of the operator are given by $a = \{a_{kj}, a_k, \tilde{a_k}, a_0\}$ and M refers to mixed boundary conditions.

We also define the elliptic operator with Dirichlet boundary condition $\mathrm{Tr}(u) = 0$ on $\partial\Omega$. It is the operator associated with the form given by the expression (2.3) with domain $D(\mathfrak{a}) = W_0^{1,2}(\Omega)$. It is a quasi-accretive and closed form and its associated operator L_a^D has the same expression as in (2.4) and subject to the Dirichlet boundary condition $\mathrm{Tr}(u) = 0$ on $\partial\Omega$.

Similarly, we define L_a^N to be the elliptic operator with Neumann type boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega.$$

It is the operator associated with the form given by the expression (2.3) with domain $D(\mathfrak{a}) = W^{1,2}(\Omega)$.
Note that L_a^D coincides with L_a^M if $\Gamma_0 = \partial\Omega$ and L_a^N coincides with L_a^M if $\Gamma_0 = \emptyset$.
Finally we define elliptic operators with Robin boundary conditions. Let $\mu \in \mathbb{R}$ be a constant and define

$$\begin{aligned}\mathfrak{a}^\mu(u,v) = &\sum_{k,j=1}^{d}\int_\Omega a_{kj}\partial_k u\overline{\partial_j v}\, dx + \sum_{k=1}^{d}\int_\Omega a_k\partial_k u\overline{v} + \tilde{a_k} u\overline{\partial_k v}\, dx + a_0 u\overline{v}\, dx \\ &- \mu\int_{\partial\Omega}\mathrm{Tr}(u)\overline{\mathrm{Tr}(v)}d\sigma \end{aligned} \tag{2.6}$$

for all $u, v \in D(\mathfrak{a}^\mu) := V$. Again, Tr denotes the trace operator. Using the standard inequality (see [2] or [3]),

$$\int_{\partial\Omega}|\mathrm{Tr}(u)|^2 \leq \varepsilon\|u\|^2_{W^{1,2}(\Omega)} + c_\varepsilon\int_\Omega |u|^2$$

which is valid for all $\varepsilon > 0$ (c_ε is a constant depending on ε) one obtains that for some positive constants w and δ

$$\mathrm{Re}\,\mathfrak{a}^\mu(u,u) + w\int_\Omega |u|^2 \geq \delta\|u\|^2_{W^{1,2}(\Omega)}.$$

From this it follows that $\mathfrak{a}^\mu$ is a quasi-accretive and closed sesquilinear form. One can associate with $\mathfrak{a}^\mu$ an operator L_a^μ. This operator has the same expression (2.4) and it is subject to the Robin boundary conditions

$$\begin{cases}\mathrm{Tr}(u) = 0 \quad \text{on } \Gamma_0 \\ \dfrac{\partial u}{\partial\nu} = \mu\ \mathrm{Tr}(u) \quad \text{on } \Gamma_1.\end{cases} \tag{2.7}$$

Actually, the boundary conditions (2.7) are mixed Robin boundary conditions in the sense that we have the Dirichlet condition on Γ_0 and the Robin one on Γ_1. For simplicity we ignore the word "mixed" and refer to (2.7) as the Robin boundary conditions.
According to our previous notation, if $\mu = 0$, then $\mathfrak{a}^0 = \mathfrak{a}$ and $L_a^0 = L_a^M$.
Note that we may choose here μ to be a bounded measurable function on the boundary rather than just a constant.

The partial Dirichlet-to-Neumann operator on $\partial\Omega$.

Suppose as before that $a = \{a_{kj}, a_k, \tilde{a_k}, a_0\}$ are bounded measurable and satisfy the ellipticity condition (2.1). Let Γ_0, Γ_1, V be as above and $\mathfrak{a}$ is the sesquilinear form defined by (2.3).
We define the space

$$V_H := \{u \in V, \mathfrak{a}(u,g) = 0 \text{ for all } g \in W_0^{1,2}(\Omega)\}. \tag{2.8}$$

Then V_H is a closed subspace of V. It is interpreted as the space of harmonic functions for the operator L_a (given by (2.4)) with the additional property that $\mathrm{Tr}(u) = 0$ on Γ_0.
We start with the following simple lemma.

Lemma 2.1. *Suppose that $0 \notin \sigma(L_a^D)$. Then*

$$V = V_H \oplus W_0^{1,2}(\Omega). \tag{2.9}$$

Proof. We argue as in [11], Section 2 or [2]. Let us denote by $\mathfrak{a}^D$ the form associated with L_a^D, that is, $\mathfrak{a}^D$ is given by (2.3) with $D(\mathfrak{a}^D) = W_0^{1,2}(\Omega)$. There exists an operator $\mathcal{L}_a^D : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega) := W_0^{1,2}(\Omega)'$ (the anti-dual of $W_0^{1,2}(\Omega)$) associated with $\mathfrak{a}^D$ in the sense

$$\langle \mathcal{L}_a^D h, g\rangle = \mathfrak{a}^D(h,g)$$

for all $h, g \in W_0^{1,2}(\Omega)$. The notation $\langle \cdot, \cdot \rangle$ denotes the duality $W_0^{1,2}(\Omega)' - W_0^{1,2}(\Omega)$. Since $0 \notin \sigma(L_a^D)$, then L_a^D is invertible. Therefore $\mathcal{L}_a^D$, seen as operator on $W_0^{1,2}(\Omega)'$ with domain $W_0^{1,2}(\Omega)$, is also invertible on $W_0^{1,2}(\Omega)'$ since the two operators L_a^D and $\mathcal{L}_a^D$ have the same spectrum (see e.g., [1], Proposition 3.10.3). Now we fix $u \in V$ and consider the (anti-)linear functional

$$F : v \mapsto \mathfrak{a}(u,v).$$

Clearly, $F \in W_0^{1,2}(\Omega)'$ and hence there exists a unique $u_0 \in W_0^{1,2}(\Omega)$ such that $\mathcal{L}_a^D u_0 = F$, i.e., $\langle \mathcal{L}_a^D u_0, g\rangle = F(g)$ for all $g \in W_0^{1,2}(\Omega)$. This means that $\mathfrak{a}(u - u_0, g) = 0$ for all $g \in W_0^{1,2}(\Omega)$ and hence $u - u_0 \in V_H$. Thus, $u = u - u_0 + u_0 \in V_H + W_0^{1,2}(\Omega)$. Finally, if $u \in V_H \cap W_0^{1,2}(\Omega)$ then $\mathfrak{a}(u,g) = 0$ for all $g \in W_0^{1,2}(\Omega)$. This means that $u \in D(L_a^D)$ with $L_a^D u = 0$. Since L_a^D is invertible we conclude that $u = 0$. ☐

As a consequence of Lemma 2.1, the trace operator $\mathrm{Tr} : V_H \to L^2(\partial\Omega)$ is injective and

$$\mathrm{Tr}(V_H) = \mathrm{Tr}(V). \tag{2.10}$$

In the rest of this section we assume that $0 \notin \sigma(L_a^D)$. We define on $L^2(\partial\Omega, d\sigma)$ the sesquilinear form

$$\mathfrak{b}(\varphi, \psi) := \mathfrak{a}(u, v) \tag{2.11}$$

where $u, v \in V_H$ are such that $\varphi = \mathrm{Tr}(u)$ and $\psi = \mathrm{Tr}(v)$. This means that $D(\mathfrak{b}) = \mathrm{Tr}(V_H)$ and by (2.10)

$$D(\mathfrak{b}) = \mathrm{Tr}(V_H) = \mathrm{Tr}(V). \tag{2.12}$$

Lemma 2.2. *There exist positive constants w, δ and M such that*

$$\mathrm{Re}\,\mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2 \geq \delta \|u\|^2_{W^{1,2}(\Omega)} \tag{2.13}$$

and

$$|\mathfrak{b}(\varphi, \psi)| \leq M \left[\mathrm{Re}\,\mathfrak{b}(\varphi, \varphi) + w \int_{\partial\Omega} |\varphi|^2\right]^{1/2} \left[\mathrm{Re}\,\mathfrak{b}(\psi, \psi) + w \int_{\partial\Omega} |\psi|^2\right]^{1/2} \tag{2.14}$$

for all $\varphi, \psi \in D(\mathfrak{b})$. In the first inequality, $u \in V_H$ is such that $Tr(u) = \varphi$.

Proof. It is well known that $\mathrm{Tr} : W^{1,2}(\Omega) \to L^2(\partial\Omega)$ is a compact operator and since $\mathrm{Tr} : V_H \to L^2(\partial\Omega)$ is injective it follows that for every $\epsilon > 0$ there exists a constant $c > 0$ such that

$$\int_\Omega |u|^2 \leq \epsilon \|u\|^2_{W^{1,2}} + c \int_{\partial\Omega} |\mathrm{Tr}(u)|^2 \tag{2.15}$$

for all $u \in V_H$ (see, e.g., [2]). In particular,

$$\int_\Omega |u|^2 \leq \frac{\epsilon}{1-\epsilon} \int_\Omega |\nabla u|^2 + \frac{c}{1-\epsilon} \int_{\partial\Omega} |\varphi|^2. \tag{2.16}$$

Now, let $\varphi \in D(\mathfrak{b}) = \mathrm{Tr}(V_H)$ and $u \in V_H$ such that $\varphi = \mathrm{Tr}(u)$. It follows from the ellipticity assumption (2.1) and the boundedness of the coefficients that for some constant $c_0 > 0$

$$\mathrm{Re}\,\mathfrak{a}(u, u) \geq \frac{\eta}{2} \int_\Omega |\nabla\, u|^2 - c_0 \int_\Omega |u|^2.$$

Therefore, using (2.16) and the definition of $\mathfrak{b}$ we obtain

$$\begin{aligned} \mathrm{Re}\,\mathfrak{b}(\varphi, \varphi) &= \mathrm{Re}\,\mathfrak{a}(u, u) \\ &\geq \left(\frac{\eta}{2} - \frac{c_0\epsilon}{1-\epsilon}\right) \int_\Omega |\nabla\, u|^2 - \frac{cc_0}{1-\epsilon} \int_{\partial\Omega} |\varphi|^2. \end{aligned}$$

Taking $\epsilon > 0$ small enough we obtain (2.13).

In order to prove the second inequality, we use the definition of $\mathfrak{b}$ and again the boundedness of the coefficients to see that

$$\begin{aligned} |\mathfrak{b}(\varphi, \psi)| &= |\mathfrak{a}(u, v)| \\ &\leq C\|u\|_{W^{1,2}} \|v\|_{W^{1,2}}. \end{aligned}$$

Thus, (2.14) follows from (2.13). ∎

Corollary 2.3. *The form $\mathfrak{b}$ is continuous, quasi-accretive and closed.*

Proof. Continuity of $\mathfrak{b}$ is exactly (2.14). Quasi-accretivity means that

$$\operatorname{Re}\mathfrak{b}(\varphi,\varphi) + w\int_{\partial\Omega} |\varphi|^2 \geq 0$$

for some w and all $\varphi \in D(\mathfrak{b})$. This follows from (2.13).
Now we prove that $\mathfrak{b}$ is closed which means that $D(\mathfrak{b})$ is complete for the norm

$$\|\varphi\|_{\mathfrak{b}} := \left(\operatorname{Re}\mathfrak{b}(\varphi,\varphi) + w\int_{\partial\Omega} |\varphi|^2\right)^{1/2}$$

in which w is as in (2.13). If (φ_n) is a Cauchy sequence for $\|\cdot\|_{\mathfrak{b}}$ then by (2.13) the corresponding $(u_n) \in V_H$ with $\operatorname{Tr}(u_n) = \varphi_n$ is a Cauchy sequence in V_H. Since V_H is a closed subspace of V it follows that u_n is convergent to some u in V_H. Set $\varphi := \operatorname{Tr}(u)$. We have $\varphi \in D(\mathfrak{b})$ and the definition of $\mathfrak{b}$ together with continuity of Tr as an operator from $W^{1,2}(\Omega)$ to $L^2(\partial\Omega)$ show that φ_n converges to φ for the norm $\|\cdot\|_{\mathfrak{b}}$. This means that $\mathfrak{b}$ is a closed form. $\square$

Note that the domain $\operatorname{Tr}(V_H)$ of $\mathfrak{b}$ may not be dense in $L^2(\partial\Omega)$ since functions in this domain vanish on Γ_0. Indeed,

$$H := \overline{D(\mathfrak{b})}^{L^2(\partial\Omega)} = L^2(\Gamma_1) \oplus \{0\}. \tag{2.17}$$

The direct inclusion follows from the fact that if $\varphi_n \in D(\mathfrak{b})$ converges in $L^2(\partial\Omega)$ then after extracting a subsequence we have a.e. convergence. Since $\varphi_n = 0$ on Γ_0 we obtain that the limit $\varphi = 0$ on Γ_0. The reverse inclusion can be proved as follows. Let Γ_2 be a closed subset of $\mathbb{R}^d$ with $\Gamma_2 \subset \Gamma_1$ and consider the space $E = \{u_{|\Gamma_2} : u \in W^{1,\infty}(\mathbb{R}^d), u_{|\Gamma_0} = 0\}$. Then $E \subset C(\Gamma_2)$ and an easy application of the Stone-Weierstrass theorem shows that E is dense in $C(\Gamma_2)$. Now given $\varphi \in C_c(\Gamma_1)$ and $\epsilon > 0$ we find Γ_2 such that $\|\mathbb{1}_{\Gamma_1\setminus\Gamma_2}\|_2 < \epsilon$ and $u_{|\Gamma_2} \in E$ such that $\|u_{|\Gamma_2} - \varphi\|_{C(\Gamma_2)} < \epsilon$. Finally we take $\chi \in C_c^\infty(\mathbb{R}^d)$ such that $\chi = 1$ on Γ_2. Then $(u\chi)_{|\Omega} \in V$ and

$$\begin{aligned} \|u\chi - \varphi\|_{L^2(\Gamma_1)} &\leq \|u - \varphi\|_{L^2(\Gamma_2)} + \|\chi\|_{L^2(\Gamma_1\setminus\Gamma_2)} \\ &\leq \epsilon|\Gamma_2| + \|\chi\|_\infty \epsilon. \end{aligned}$$

Here $|\Gamma_2|$ denotes the measure of Γ_2. These inequalities together with the fact that $C_c(\Gamma_1)$ is dense in $L^2(\Gamma_1)$ imply (2.17).

We return to the form $\mathfrak{b}$ defined above. We associate with $\mathfrak{b}$ an operator $\mathcal{N}_{\Gamma_1}$. It is defined by

$$D(\mathcal{N}_{\Gamma_1}) := \{\varphi \in D(\mathfrak{b}), \exists\psi \in H : \mathfrak{b}(\varphi,\xi) = \int_{\Gamma_1} \psi\overline{\xi} \ \forall\xi \in D(\mathfrak{b})\}, \quad \mathcal{N}_{\Gamma_1}\varphi = \psi.$$

The operator $\mathcal{N}_{\Gamma_1}$ can be interpreted as an operator on $L^2(\partial\Omega)$ defined as follows: if $\varphi \in D(\mathcal{N}_{\Gamma_1})$ then there exists a unique $u \in V_H$ such that $\varphi = \operatorname{Tr}\,(u)$ and

$$\varphi_{|\Gamma_0} = 0, \quad \mathcal{N}_{\Gamma_1}(\varphi) = \frac{\partial u}{\partial \nu} \text{ on } \Gamma_1. \tag{2.18}$$

Again $\frac{\partial u}{\partial \nu}$ is interpreted in the weak sense as the conormal derivative that is $\sum_{j=1}^{d}\left(\sum_{k=1}^{d} a_{kj}\partial_k u + \tilde{a_j}\varphi\right)\nu_j$. In the particular case where $a_{kj} = \delta_{kj}$ and $a_1 = \cdots = a_d = 0$ the right hand side of (2.18) is seen as the normal derivative on the boundary. All this can be made precise by applying the Green formula if the boundary and the coefficients are smooth enough.
We call $\mathcal{N}_{\Gamma_1}$ the *partial Dirichlet-to-Neumann* operator on $L^2(\partial\Omega)$ or the *Dirichlet-to-Neumann operator with partial data.* The term *partial* refers to the fact that $\mathcal{N}_{\Gamma_1}$ is known only on the part Γ_1 of the boundary $\partial\Omega$.
It follows from the general theory of forms that $-\mathcal{N}_{\Gamma_1}$ generates a holomorphic semigroup $e^{-t\mathcal{N}_{\Gamma_1}}$ on H. We define $T_t^{\Gamma_1}$ on $L^2(\partial\Omega)$ by

$$T_t^{\Gamma_1}\varphi = e^{-t\mathcal{N}_{\Gamma_1}}(\varphi \mathbb{1}_{\Gamma_1}) \oplus 0.$$

We shall refer to $(T_t^{\Gamma_1})_{t\geq 0}$ as the "semigroup" generated by $-\mathcal{N}_{\Gamma_1}$ on $L^2(\partial\Omega)$. It is clear that

$$\|T_t^{\Gamma_1}\|_{\mathcal{L}(L^2(\partial\Omega))} \leq e^{-w_0 t}, \quad t \geq 0, \tag{2.19}$$

for some constant w_0. Note that if the form $\mathfrak{a}$ is symmetric, then $\mathfrak{b}$ is also symmetric and hence $\mathcal{N}_{\Gamma_1}$ is self-adjoint. In this case, (2.19) holds with $w_0 = \inf \sigma(\mathcal{N}_{\Gamma_1})$ which also coincides with the first eigenvalue of $\mathcal{N}_{\Gamma_1}$. For all this, see e.g. [23], Chapter 1.

3 Positivity and domination

In this section we study some properties of the semigroup $(T_t^{\Gamma_1})_{t\geq 0}$. We assume throughout this section that

$$a_{jk} = a_{kj},\, \tilde{a_k} = a_k, a_0 \in L^\infty(\Omega, \mathbb{R}). \tag{3.1}$$

We recall that L_a^D is the elliptic operator with Dirichlet boundary conditions defined in the previous section. Its associated symmetric form $\mathfrak{a}^D$ is given by (2.3) and has domain $W_0^{1,2}(\Omega)$. We shall need the accretivity assumption of $\mathfrak{a}^D$ (or equivalently the self-adjoint operator L_a^D is non-negative) which means that

$$\mathfrak{a}^D(u,u) \geq 0 \text{ for all } u \in W_0^{1,2}(\Omega). \tag{3.2}$$

Theorem 3.1. *Suppose that* $0 \notin \sigma(L_a^D)$, (3.1) *and that* L_a^D *is accretive.*
a) The semigroup $(T_t^{\Gamma_1})_{t\geq 0}$ *is positive (i.e., it maps non-negative functions of* $L^2(\partial\Omega)$ *into non-negative functions).*
b) Suppose in addition that $a_0 \geq 0$ *and* $a_k = 0$ *for all* $k \in \{1, \cdots, d\}$. *Then* $(T_t^{\Gamma_1})_{t\geq 0}$ *is a sub-Markovian semigroup.*

Recall that the sub-Markovian property means that for $\varphi \in L^2(\partial\Omega)$ and $t \geq 0$

$$0 \leq \varphi \leq 1 \Rightarrow 0 \leq T_t^{\Gamma_1}\varphi \leq 1.$$

This property implies in particular that $(T_t^{\Gamma_1})_{t\geq 0}$ extends from $L^2(\partial\Omega)$ to $L^p(\partial\Omega)$ for all $p \in [2, \infty[$. Since $\mathfrak{a}$ is symmetric then so is $\mathfrak{b}$ and one obtains by duality that $(T_t^{\Gamma_1})_{t\geq 0}$ extends also to $L^p(\partial\Omega)$ for $p \in [1, 2]$.

Proof. The proof follows exactly the same lines as for Theorem 2.3 in [11].
a) By the well known Beurling–Deny criteria (see [9], Section 1.3 or [23], Theorem 2.6), it suffices to prove that $\varphi^+ \in D(\mathfrak{b})$ and $\mathfrak{b}(\varphi^+, \varphi^-) \leq 0$ for all real-valued $\varphi \in D(\mathfrak{b})$. Note that the fact that $D(\mathfrak{b})$ is not densely defined does not affect the the statements of the Beurling-Deny criteria.
Let $\varphi \in D(\mathfrak{b})$ be real-valued. There exists a real-valued $u \in H_V$ such that $\varphi = \mathrm{Tr}(u)$. Then $\varphi^+ = \mathrm{Tr}(u^+) \in \mathrm{Tr}(V) = \mathrm{Tr} H_V = D(\mathfrak{b})$. This follows from the fact that $v^+ \in V$ for all $v \in V$ (see [23], Section 4.2).
By Lemma 2.1 we can write $u^+ = u_0 + u_1$ and $u^- = v_0 + v_1$ with $u_0, v_0 \in W_0^{1,2}(\Omega)$ and $u_1, v_1 \in H_V$. Hence, $u = u^+ - u^- = (u_0 - v_0) + (u_1 - v_1)$. Since $u, u_1 - v_1 \in H_V$ it follows that $u_0 = v_0$. Therefore,

$$\begin{aligned}
\mathfrak{b}(\varphi^+, \varphi^-) &= \mathfrak{a}(u_1, v_1) = \mathfrak{a}(u_1, v_0 + v_1) = \mathfrak{a}(u_0 + u_1, v_0 + v_1) - \mathfrak{a}(u_0, v_0 + v_1) \\
&= \mathfrak{a}(u^+, u^-) - \mathfrak{a}(u_0, v_0) = -\mathfrak{a}(u_0, v_0) \\
&= -\mathfrak{a}(u_0, u_0) = -\mathfrak{a}^D(u_0, u_0).
\end{aligned}$$

Here we use the fact that

$$\begin{aligned}
\mathfrak{a}(u^+, u^-) = \sum_{k,j=1}^{d} \int_\Omega a_{kj} \partial_k(u^+) \partial_j(u^-) + \sum_{k=1}^{d} \int_\Omega a_k \partial_k u^+ \, u^- + a_k u^+ \partial_k u^- \\
+ \int_\Omega a_0 u^+ \, u^- = 0.
\end{aligned}$$

By assumption (3.2) we have $\mathfrak{a}^D(u_0, u_0) \geq 0$ and we obtain $\mathfrak{b}(\varphi^+, \varphi^-) \leq 0$. This proves the positivity of $(T_t^{\Gamma_1})_{t \geq 0}$ on $L^2(\partial\Omega)$.

b) By [22] or [23], Corollary 2.17 it suffices to prove that $\mathbb{1} \wedge \varphi := \inf(\mathbb{1}, \varphi) \in D(\mathfrak{b})$ and $\mathfrak{b}(\mathbb{1} \wedge \varphi, (\varphi - \mathbb{1})^+) \geq 0$ for all $\varphi \in D(\mathfrak{b})$ with $\varphi \geq 0$. Let $\varphi \in D(\mathfrak{b})$ and suppose that $\varphi \geq 0$. Let $u \in H_V$ be real-valued such that $\varphi = \mathrm{Tr}(u)$. Note that $\mathbb{1} \wedge u \in V$ (see [23], Section 4.3). We decompose $\mathbb{1} \wedge u = u_0 + u_1 \in W_0^{1,2}(\Omega) \oplus H_V$. Then

$$(u - \mathbb{1})^+ = u - \mathbb{1} \wedge u = (-u_0) + (u - u_1) \in W_0^{1,2}(\Omega) \oplus H_V.$$

Therefore,

$$\begin{aligned}
\mathfrak{b}(\mathbb{1} \wedge \varphi, (\varphi - \mathbb{1})^+) &= \mathfrak{a}(u_1, u - u_1) = \mathfrak{a}(u_0 + u_1, u - u_1) \\
&= \mathfrak{a}(u_0 + u_1, -u_0 + u - u_1) + \mathfrak{a}(u_0 + u_1, u_0) \\
&= \mathfrak{a}(u_0 + u_1, -u_0 + u - u_1) + \mathfrak{a}(u_0, u_0) \\
&= \sum_{k,j=1}^{d} \int_\Omega a_{kj} \partial_k(\mathbb{1} \wedge u) \partial_j((u - \mathbb{1})^+) + \\
&\quad \int_\Omega a_0 (\mathbb{1} \wedge u)(u - \mathbb{1})^+ + \mathfrak{a}^D(u_0, u_0) \\
&= \int_\Omega a_0 \, (u - \mathbb{1})^+ + \mathfrak{a}^D(u_0, u_0) \geq 0.
\end{aligned}$$

This proves that $\mathfrak{b}(\mathbb{1} \wedge \varphi, (\varphi - \mathbb{1})^+) \geq 0$. ∎

Next we have the following domination property.

Theorem 3.2. *Suppose that a_{kj}, a_k, $\tilde{a_k}$ and a_0 satisfy* (3.1). *Suppose also that L_a^D is accretive with $0 \notin \sigma(L_a^D)$. Let Γ_0 and $\tilde{\Gamma_0}$ be two closed subsets of the boundary such that $\Gamma_0 \subseteq \tilde{\Gamma_0}$. Then for every $0 \leq \varphi \in L^2(\partial\Omega)$*

$$0 \leq T_t^{\tilde{\Gamma_1}} \varphi \leq T_t^{\Gamma_1} \varphi.$$

Proof. Let $\tilde{\Gamma_1}$ be the complement of $\tilde{\Gamma_0}$ in $\partial\Omega$. Denote by $\mathfrak{b}$ and $\tilde{\mathfrak{b}}$ the sesquilinear forms associated with $\mathcal{N}_{\Gamma_1}$ and $\mathcal{N}_{\tilde{\Gamma_1}}$, respectively. Clearly, $\tilde{\mathfrak{b}}$ is a restriction of $\mathfrak{b}$ and hence it is enough to prove that $D(\tilde{\mathfrak{b}})$ is an ideal of $D(\mathfrak{b})$ and apply [22] or [23], Theorem 2.24. For this, let $0 \leq \varphi \leq \psi$ with $\varphi \in D(\mathfrak{b})$ and $\psi \in D(\tilde{\mathfrak{b}})$. This means that φ and ψ are respectively the traces on $\partial\Omega$ of $u, v \in W^{1,2}(\Omega)$ such that

$$\varphi = \mathrm{Tr}(u) = 0 \text{ on } \Gamma_0 \quad \text{and } \psi = \mathrm{Tr}(v) = 0 \text{ on } \tilde{\Gamma_0}.$$

Since $0 \leq \varphi \leq \psi$ we have $\varphi = 0$ on $\tilde{\Gamma_0}$. This equality gives $\varphi \in D(\tilde{\mathfrak{b}})$ and this shows that $D(\tilde{\mathfrak{b}})$ is an ideal of $D(\mathfrak{b})$. ☐

The next result shows monotonicity with respect to the potential a_0. This was already proved in [11] Theorem 2.4, in the case where $L_a^D = -\Delta + a_0$. The proof given there works also in the general framework of the present paper.

As above let a_{kj}, a_k and a_0 be real-valued and let $(T_t^{\Gamma_1, a_0})_{t\geq 0}$ denote the semigroup $(T_t^{\Gamma_1})_{t\geq 0}$ defined above. Suppose that b_0 is a real-valued function and denote by $(T_t^{\Gamma_1, b_0})_{t\geq 0}$ be the semigroup of $\mathcal{N}_{\Gamma_1}$ with coefficients a_{kj}, a_k and b_0 (i.e. a_0 is replaced by b_0). Then we have

Theorem 3.3. *Suppose that a_{kj}, a_k, $\tilde{a_k}$ and a_0 satisfy* (3.1). *Suppose again that $0 \notin \sigma(L_a^D)$ and L_a^D is accretive. If $a_0 \leq b_0$ then*

$$0 \leq T_t^{\Gamma_1, b_0} \varphi \leq T_t^{\Gamma_1, a_0} \varphi$$

for all $0 \leq \varphi \in L^2(\partial\Omega)$ and $t \geq 0$.

4 Proof of the main result

In this section we prove Theorem 1.1. We recall briefly the operators introduced in Section 2.

For $\mu \in \mathbb{R}$ and recall the operator L_a^μ associated with the form $\mathfrak{a}^\mu$ given by (2.6) with domain $D(\mathfrak{a}^\mu) := V$ and V is again given by (2.2). The operator associated with $\mathfrak{a}^\mu$ is L_a^μ. It is given by the formal expression (2.4) and it is subject to mixed and Robin boundary conditions (2.7).
We also recall that L_a^D is the operator subject to the Dirichlet boundary conditions and L_a^M is subject to mixed boundary conditions.

Fix $\lambda \notin \sigma(L_a^D)$. We denote by $\mathcal{N}_{\Gamma_1, a}(\lambda)$ the partial D-t-N operator with the coefficients $\{a_{kj}, a_k, a_0 - \lambda\}$. It is the operator associated with the form

$$\mathfrak{b}(\varphi, \psi) := \sum_{k,j}^{d} \int_\Omega a_{kj} \partial_k u \overline{\partial_j v} \, dx + \sum_{k=1}^{d} \int_\Omega a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} \, dx + (a_0 - \lambda) u \overline{v} \, dx$$

where $u, v \in V_H(\lambda)$ with $\mathrm{Tr}(u) = \varphi$, $\mathrm{Tr}(v) = \psi$ and

$$V_H(\lambda) := \{u \in V, \mathfrak{a}(u, g) = \lambda \int_\Omega u \overline{g} \text{ for all } g \in W_0^{1,2}(\Omega)\}, \tag{4.1}$$

This space is the same as in (2.8) but now with a_0 replaced by $a_0 - \lambda$.

We restate the main theorem using the notation introduced in Section 2.

Theorem 4.1. *Suppose that Ω is a bounded Lipchitz domain of $\mathbb{R}^d$ with $d \geq 2$. Let Γ_0 be a closed subset of $\partial\Omega$, $\Gamma_0 \neq \partial\Omega$ and $\Gamma_1 = \partial\Omega \setminus \Gamma_0$. Let $a = \{a_{kj} = \overline{a_{jk}}, a_k = \overline{\tilde{a_k}}, a_0 = \overline{a_0}\}$ and $b = \{b_{kj} = \overline{b_{jk}}, b_k = \overline{\tilde{b_k}}, b_0 = \overline{b_0}\}$ be bounded measurable functions on Ω such that a_{kj} and b_{kj} satisfy the ellipticity condition (2.1). If $d \geq 3$ we assume in addition that the coefficients a_{kj}, b_{kj}, a_k and b_k are Lipschitz continuous on $\overline{\Omega}$.*
Suppose that $\mathcal{N}_{\Gamma_1,a}(\lambda) = \mathcal{N}_{\Gamma_1,b}(\lambda)$ for all λ in a set having an accumulation point in $\rho(L_a^D) \cap \rho(L_b^D)$. Then:
i) The operators L_a^μ and L_b^μ are unitarily equivalent for all $\mu \in \mathbb{R}$.
ii) The operators L_a^M and L_b^M are unitarily equivalent.
iii) The operators L_a^D and L_b^D are unitarily equivalent.

Moreover, for every $\lambda \in \sigma(L_a^\mu) = \sigma(L_b^\mu)$ with $\lambda \notin \sigma(L_a^D) = \sigma(L_b^D)$, the sets $\{Tr(u), u \in Ker(\lambda I - L_a^\mu)\}$ and $\{Tr(v), v \in Ker(\lambda I - L_b^\mu)\}$ coincide. The same property holds for the operators L_a^M and L_b^M.

We shall need several preparatory results. We start with the following theorem which was proved in [2] and [3] in the case where $a_{kj} = \delta_{kj}$, $a_k = 0$, a_0 is a constant and $\Gamma_1 = \partial\Omega$.

Theorem 4.2. *Let $a = \{a_{kj} = \overline{a_{jk}}, a_k = \overline{\tilde{a_k}}, a_0 = \overline{a_0}\}$ be bounded measurable functions on Ω such that a_{kj} satisfy the ellipticity condition (2.1).*
Let $\mu, \lambda \in \mathbb{R}$ and $\lambda \notin \sigma(L_a^D)$. Then:
1) $\mu \in \sigma(\mathcal{N}_{\Gamma_1,a}(\lambda)) \Leftrightarrow \lambda \in \sigma(L_a^\mu)$. In addition, if $u \in Ker(\lambda - L_a^\mu)$, $u \neq 0$ then $\varphi := Tr(u) \in Ker(\mu - \mathcal{N}_{\Gamma_1,a}(\lambda))$ and $\varphi \neq 0$. Conversely, if $\varphi \in Ker(\mu - \mathcal{N}_{\Gamma_1,a}(\lambda))$, $\varphi \neq 0$, then there exists $u \in Ker(\lambda - L_a^\mu)$, $u \neq 0$ such that $\varphi = Tr(u)$.
2) $\dim Ker(\mu - \mathcal{N}_{\Gamma_1,a}(\lambda)) = \dim Ker(\lambda - L_a^\mu)$.

Proof. We follow a similar idea as in [2] and [3]. It is enough to prove that the mapping

$$S : \mathrm{Ker}(\lambda - L_a^\mu) \to \mathrm{Ker}(\mu - \mathcal{N}_{\Gamma_1,a}(\lambda)),\ u \mapsto \mathrm{Tr}(u)$$

is an isomorphism. First, we prove that S is well defined. Let $u \in \mathrm{Ker}(\lambda - L_a^\mu)$. Then $u \in D(L_a^\mu)$ and $L_a^\mu u = \lambda u$. By the definition of L_a^μ we have $u \in V$ and for all $v \in V$

$$\begin{aligned}\sum_{k,j=1}^{d} \int_\Omega a_{kj}\partial_k u\overline{\partial_j v} + \sum_{k=1}^{d} \int_\Omega a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} \\ + \int_\Omega a_0 u\overline{v} - \lambda \int_\Omega u\overline{v} = \mu \int_{\partial\Omega} \mathrm{Tr}(u)\overline{\mathrm{Tr}(v)}. \end{aligned} \tag{4.2}$$

Taking $v \in W_0^{1,2}(\Omega)$ yields $u \in V_H(\lambda)$. Note that (4.2) also holds for $v \in V_H(\lambda)$. Hence it follows from the definition of $\mathcal{N}_{\Gamma_1,a}(\lambda)$ that

$$\varphi := \mathrm{Tr}(u) \in D(\mathcal{N}_{\Gamma_1,a}(\lambda)) \text{ and } \mathcal{N}_{\Gamma_1,a}(\lambda)\varphi = \mu\varphi.$$

This means that $S(u) \in \mathrm{Ker}(\mu - \mathcal{N}_{\Gamma_1,a}(\lambda))$.

Suppose now that $u \in \mathrm{Ker}(\lambda - L_a^\mu)$ with $u \neq 0$. If $S(u) = 0$ then $u \in W_0^{1,2}(\Omega)$. Therefore, it follows from (4.2) that for all $v \in V$

$$\sum_{k,j=1}^{d} \int_\Omega a_{kj}\partial_k u\overline{\partial_j v} + \sum_{k=1}^{d} \int_\Omega a_k \partial_k u \overline{v} + \overline{a_k} u \overline{\partial_k v} + \int_\Omega (a_0 - \lambda) u\overline{v} = 0. \tag{4.3}$$

This implies that $u \in V_H(\lambda)$. We conclude by Lemma 2.1 that $u = 0$. Thus S is injective.
We prove that S is surjective. Let $\varphi \in \mathrm{Ker}(\mu - \mathcal{N}_{\Gamma_1,a}(\lambda))$. Then by the definition of $\mathcal{N}_{\Gamma_1,a}(\lambda)$, there exists $u \in V_H(\lambda)$ such that $\varphi = \mathrm{Tr}(u)$ and u satisfies (4.2) for all $v \in V_H(\lambda)$. If $v \in V$ we write $v = v_0 + v_1 \in W_0^{1,2}(\Omega) \oplus V_H(\lambda)$ and see that (4.2) holds for u and v. This means that $u \in D(L_a^\mu)$ and $L_a^\mu u = \lambda u$. $\square$

Lemma 4.3. *For $\lambda \in \mathbb{R}$ large enough, $(\lambda + L_a^\mu)^{-1}$ converges in $\mathcal{L}(L^2(\Omega))$ to $(\lambda + L_a^D)^{-1}$ as $\mu \to -\infty$.*

This is Proposition 2.6 in [2] when $a_{kj} = \delta_{kj},\ a_k = a_0 = 0$. The proof given in [2] remains valid in our setting. Note that the idea of proving the uniform convergence here is based on a criterion from [8] (see Appendix B) which states that it is enough to check that for all $(f_n), f \in L^2(\Omega)$

$$f_n \rightharpoonup f \Rightarrow (\lambda + L_a^{\mu_n})^{-1} f_n \to (\lambda + L_a^D)^{-1} f, \tag{4.4}$$

for every sequence $\mu_n \to -\infty$. The first convergence is in the weak sense in $L^2(\Omega)$ and the second one is the strong convergence. It is not difficult to check (4.4).

From now on, we denote by $(\lambda_{a,n}^\mu)_{n\geq 1}$ the eigenvalues of L_a^μ, repeated according to their multiplicities. We have for each $\mu \in \mathbb{R}$

$$\lambda_{a,1}^\mu \leq \lambda_{a,2}^\mu \leq \cdots \to +\infty.$$

Similarly for the eigenvalues $(\lambda_{a,n}^D)_{n\geq 1}$ of L_a^D. These eigenvalues satisfy the standard min-max principle since the operators L_a^μ and L_a^D are self-adjoint by our assumptions.

A well known consequence of the previous lemma is that the spectrum of L_a^μ converges to the spectrum of L_a^D. More precisely, for all k,

$$\lambda_{a,k}^\mu \to \lambda_{a,k}^D \text{ as } \mu \to -\infty. \tag{4.5}$$

In addition, we have the following lemma which will play a fundamental role.

Lemma 4.4. *Let $a = \{a_{kj} = \overline{a_{jk}}, a_k = \overline{\tilde{a_k}}, a_0 = \overline{a_0}\}$ be bounded measurable functions on Ω such that a_{kj} satisfy the ellipticity condition (2.1). If $d \geq 3$ we assume in addition that the coefficients a_{kj} and a_k are Lipschitz continuous on $\overline{\Omega}$. Then for each k, $\mu \mapsto \lambda_{a,k}^\mu$ is strictly decreasing on $\mathbb{R}$ and $\lambda_{a,k} \to -\infty$ as $\mu \to +\infty$.*

Proof. Firstly, by the min-max principle $\lambda_{a,k}^\mu \leq \lambda_{a,k}^D$ and the function $\mu \mapsto \lambda_{a,k}^\mu$ is non-increasing. Fix $k \geq 0$ and suppose that $\mu \mapsto \lambda_{a,k}^\mu$ is constant on $[\alpha, \beta]$ for some $\alpha < \beta$. For each μ we take a normalized eigenvector u^μ such that $\mathrm{Tr}(u^{\mu+h}) \to \mathrm{Tr}(u^\mu)$ in $L^2(\partial\Omega)$ as $h \to 0$ (or as $h_n \to 0$ for some sequence h_n). Indeed, due to regularity properties $\mu \mapsto \lambda_{a,k}^\mu$ is continuous (see [17], Chapter VII) and hence $(\lambda_{a,k}^{\mu+h})_h$ is bounded for small h. The equality $\mathfrak{a}^{\mu+h}(u^{\mu+h}, u^{\mu+h}) = \lambda_{a,k}^{\mu+h}$ implies that $\mathfrak{a}^{\mu+h}(u^{\mu+h}, u^{\mu+h})$ is bounded w.r.t. h (for small h). This latter property and ellipticity easily imply that $(u^{\mu+h})_h$ is bounded in V. After extracting a sequence we may assume that $(u^{\mu+h})_h$ converges weakly in V to some u as $h \to 0$. The compactness embedding of V

in $L^2(\Omega)$ as well as the compactness of the trace operator show that $(u^{\mu+h})_h$ converges to u in $L^2(\Omega)$ and $\mathrm{Tr}(u^{\mu+h})$ converges to $\mathrm{Tr}(u)$ in $L^2(\partial\Omega)$. On the other hand for every $v \in V$, the equality

$$\mathfrak{a}^{\mu+h}(u^{\mu+h}, v) = \lambda^{\mu+h}_{a,k} \int_\Omega u^{\mu+h} v \; dx$$

shows that the limit u is a normalized eigenvector of L^μ_a for the eigenvalue $\lambda^\mu_{a,k}$. We take $u^\mu := u$ and obtain the claim stated above.

Observe that

$$\int_{\Gamma_1} \mathrm{Tr}(u^{\mu+h})\overline{\mathrm{Tr}(u^\mu)} \; d\sigma = 0 \tag{4.6}$$

for all $h \neq 0$ and $\mu, \mu+h \in [\alpha, \beta]$. Indeed, using the definition of the form $\mathfrak{a}^\mu$ (see (2.6)) we have

$$\begin{aligned}
\lambda \int_\Omega u^{\mu+h}\overline{u^\mu} \; dx &= \mathfrak{a}^{\mu+h}(u^{\mu+h}, u^\mu) \\
&= \mathfrak{a}^{\mu}(u^{\mu+h}, u^\mu) - h \int_{\Gamma_1} \mathrm{Tr}(u^{\mu+h})\overline{\mathrm{Tr}(u^\mu)} \; d\sigma \\
&= \lambda \int_\Omega u^{\mu+h}\overline{u^\mu} \; dx - h \int_{\Gamma_1} \mathrm{Tr}(u^{\mu+h})\overline{\mathrm{Tr}(u^\mu)} \; d\sigma.
\end{aligned}$$

This gives (4.6). Now, letting $h \to 0$ we obtain from (4.6) and the fact that $\mathrm{Tr}(u^{\mu+h})$ converges to $\mathrm{Tr}(u^\mu)$ as $h \to 0$ that $\mathrm{Tr}(u^\mu) = 0$ on Γ_1 for all $\mu \in [\alpha, \beta]$. Hence $\mathrm{Tr}(u^\mu) = 0$ on $\partial\Omega$ since $u^\mu \in V$. Hence, L^μ has an eigenfunction $u^\mu \in W^{1,2}_0(\Omega)$. Note that if $d = 2$ or if $d \geq 3$ and the coefficients a_{kj} and a_k are Lipschitz continuous on $\overline{\Omega}$, then the operator L_a has the unique continuation property (see [26] for the case $d = 2$ and [28] for $d \geq 3$). If $d \geq 3$ and hence the coefficients are Lipschitz on $\overline{\Omega}$, we apply Proposition 2.5 in [6] to conclude that $u^\mu = 0$, but this is not possible since $\|u^\mu\|_2 = 1$. If $d = 2$ we argue in a similar way. Indeed, let $\widetilde{\Omega}$ be an open subset of $\mathbb{R}^2$ containing Ω and such that $\Gamma_0 \subset \partial\widetilde{\Omega}$ and $\widetilde{\Omega} \setminus \Omega$ contains an open ball. We extend all the coefficients to bounded measurable function $\tilde{a}_{kj}, \tilde{a}_k$ and $\tilde{a}_0$ on $\widetilde{\Omega}$. In addition, $\tilde{a}_{kj} = \overline{\tilde{a}_{jk}}$ on $\widetilde{\Omega}$ and satisfy the ellipticity condition. We extend u^μ to $\tilde{u^\mu} \in W^{1,2}_0(\widetilde{\Omega})$ by 0 outside Ω. We define in $\widetilde{\Omega}$ the elliptic operator $L_{\tilde{a}}$ as previously. For $v \in C^\infty_c(\widetilde{\Omega})$ we note that $v_{|\Omega} \in V$ and hence

$$\begin{aligned}
\int_{\widetilde{\Omega}} L_{\tilde{a}}(\tilde{u^\mu})\overline{v} dx &= \mathfrak{a}^\mu(u^\mu, v_{|\Omega}) \\
&= \lambda \int_\Omega u^\mu \overline{v}_{|\Omega} = \lambda \int_{\widetilde{\Omega}} \tilde{u^\mu}\overline{v}.
\end{aligned}$$

The term $\int_{\widetilde{\Omega}} L_{\tilde{a}}(\tilde{u^\mu})\overline{v}$ is of course interpreted in the sense of the associated sesquilinear form and the first equality uses the fact that $\tilde{u^\mu}$ is 0 on $\widetilde{\Omega} \setminus \Omega$ and $u^\mu \in W^{1,2}_0(\Omega)$. Hence, $\tilde{u^\mu}$ satisfies

$$(L_{\tilde{a}} - \lambda)(\tilde{u^\mu}) = 0$$

in the weak sense on $\widetilde{\Omega}$. We conclude by the unique continuation property ([26]) that $\tilde{u^\mu} = 0$ on $\widetilde{\Omega}$ since it is 0 on an open ball contained in $\widetilde{\Omega} \setminus \Omega$. We arrive as above to a contradiction. Hence, $\mu \mapsto \lambda^\mu_{a,k}$ is strictly decreasing on $\mathbb{R}$.

It remains to prove that for any k, $\lambda_{a,k}^{\mu} \to -\infty$ as $\mu \to +\infty$. By the min-max principle

$$\lambda_1^{\mu} \leq \sum_{k,j=1}^{d} \int_{\Omega} a_{kj} \partial_k u \overline{\partial_j u} + 2 \operatorname{Re} \sum_{k=1}^{d} \int_{\Omega} a_k \partial_k u \overline{u} + \int_{\Omega} a_0 |u|^2 - \mu \int_{\Gamma_1} |\mathrm{Tr}(u)|^2$$

for every normalized $u \in V$. Taking u such that $\mathrm{Tr}(u) \neq 0$ shows that $\lambda_{a,1}^{\mu} \to -\infty$ as $\mu \to +\infty$. Suppose now that $\lambda_{a,k}^{\mu} > w$ for some $w \in \mathbb{R}$, $k > 1$ and all $\mu \in \mathbb{R}$. Taking the smallest possible k we have $\lambda_{a,j}^{\mu} \to -\infty$ as $\mu \to +\infty$ for $j = 1, \cdots, k-1$. Of course, $\lambda_{a,j}^{\mu} > w$ for all $j \geq k$ and we may choose $w \notin \sigma(L_a^D)$. Remember also that $\mu \mapsto \lambda_{a,j}^{\mu}$ is strictly decreasing for $j = 1, \cdots, k-1$. On the other hand, by Theorem 4.2 we have $\sigma(\mathcal{N}_{\Gamma_1,a}(w)) \subset \{\mu \in \mathbb{R}, \lambda_{a,j}^{\mu} = w, j = 1, \cdots, k-1\}$. Using the fact that $\lambda_{a,j}^{\mu} \to -\infty$ as $\mu \to +\infty$ and $\mu \mapsto \lambda_{a,j}^{\mu}$ is strictly decreasing for $j = 1, \cdots, k-1$ we see that we can choose w such that the set $\{\mu \in \mathbb{R}, \lambda_{a,j}^{\mu} = w, j = 1, \cdots, k-1\}$ is finite and hence $\sigma(\mathcal{N}_{\Gamma_1,a}(w))$ is finite which is not possible since $L^2(\Gamma_1)$ has infinite dimension. $\square$

Related results to Lemma 4.4 can be found in [3] (see Proposition 3) and [25]. In both papers the proofs use the unique continuation property.

We shall also need the following lemma.

Lemma 4.5. *For every $\varphi, \psi \in Tr(V)$, the mapping*

$$\lambda \mapsto \langle \mathcal{N}_{\Gamma_1,a}(\lambda)\varphi, \psi \rangle$$

is holomorphic on $\mathbb{C} \setminus \sigma(L_a^D)$.

This result is easy to prove, see Lemma 2.4 in [6].

Proof of Theorem 1.1. As above, we denote by $(\lambda_{b,n}^{\mu})_{n\geq 1}$ and $(\lambda_{b,n}^{D})_{n\geq 1}$ the eigenvalues of the self-adjoint operators L_b^{μ} and L_b^D, respectively.

It follows from Lemma 4.5 and the assumptions that $\mathcal{N}_{\Gamma_1,a}(\lambda) = \mathcal{N}_{\Gamma_1,b}(\lambda)$ for all $\lambda \in \mathbb{C} \setminus (\sigma(L_a^D) \cup \sigma(L_b^D))$.

i) We show that for all $\mu \in \mathbb{R}$

$$\sigma(L_a^{\mu}) = \sigma(L_b^{\mu}), \tag{4.7}$$

and the eigenvalues have the same multiplicity.

Fix $\mu \in \mathbb{R}$ and suppose that $\lambda = \lambda_{a,k}^{\mu} \in \sigma(L_a^{\mu}) \setminus (\sigma(L_a^D) \cup \sigma(L_b^D))$. By Theorem 4.2, $\mu \in \sigma(\mathcal{N}_{\Gamma_1,a}(\lambda)) = \sigma(\mathcal{N}_{\Gamma_1,b}(\lambda))$ and hence $\lambda \in \sigma(L_b^{\mu})$. Thus, $\lambda = \lambda_{a,k}^{\mu} = \lambda_{b,j}^{\mu}$ for some $j \geq 1$. The second assertion of Theorem 4.2 shows that $\lambda_{a,k}^{\mu}$ and $\lambda_{b,j}^{\mu}$ have the same multiplicity. In addition, $j = k$. Indeed, if $k < j$ then

$$\lambda_{b,1}^{\mu} \leq \lambda_{b,2}^{\mu} \leq \cdots \leq \lambda_{b,k}^{\mu} \leq \cdots \leq \lambda_{b,j}^{\mu} = \lambda_{a,k}^{\mu}.$$

Each $\lambda_{b,m}^{\mu}$ coincides with an eigenvalue of L_a^{μ} (with the same multiplicity) and hence $\lambda_{a,k}^{\mu}$ is (at least) the $j-$th eigenvalue of L_a^{μ} with $j > k$ which is not possible. The same argument works if $j < k$. Using Lemma 4.4 we see that for any k there exists a discrete set $J \subset \mathbb{R}$ such that $\lambda_{a,k}^{\mu} = \lambda_{b,k}^{\mu}$ for every $\mu \in \mathbb{R} \setminus J$. By continuity of $\mu \mapsto \lambda_{a,k}^{\mu}$ and $\mu \mapsto \lambda_{b,k}^{\mu}$ these two functions coincide on $\mathbb{R}$. This

proves (4.7) and also that the multiplicities of the eigenvalues $\lambda^\mu_{a,k}$ and $\lambda^\mu_{b,k}$ are the same.

The similarity property follows by a classical argument. Recall that L^μ_a and L^μ_b are self-adjoint operators with compact resolvents. It follows that here exist orthonormal bases Φ_n and Ψ_n of $L^2(\Omega)$ which are eigenfunctions of L^μ_a and L^μ_b, respectively. Define the mapping

$$\mathcal{U} : L^2(\Omega) \to L^2(\Omega),\ \Phi_n \mapsto \Psi_n.$$

Thus for $f = \sum_n (f, \Phi_n)\Phi_n \in L^2(\Omega)$, $\mathcal{U}(f) = \sum_n (f, \Phi_n)\Psi_n$. The notation (f, Φ_n) is the scalar product in $L^2(\Omega)$. Clearly,

$$\|\mathcal{U}(f)\|_2^2 = \sum_n |(f, \Phi_n)|^2 = \|f\|_2^2.$$

The mapping $\mathcal{U}$ is an isomorphism. In addition, if $L^\mu_a \Phi_n = \lambda^\mu_{a,n}\Phi_n$ then for $f \in D(L^\mu_b)$

$$\begin{aligned}
\mathcal{U}L^\mu_a\mathcal{U}^{-1}(f) &= \mathcal{U}L^\mu_a\mathcal{U}^{-1}\left(\sum_n (f, \Psi_n)\Psi_n\right)\\
&= \mathcal{U}L^\mu_a\left(\sum_n (f, \Psi_n)\Phi_n\right)\\
&= \mathcal{U}\left(\sum_n (f, \Psi_n)\lambda^\mu_{a,n}\Phi_n\right)\\
&= \sum_n (f, \lambda^\mu_{b,n}\Psi_n)\Psi_n\\
&= L^\mu_b(f).
\end{aligned}$$

Thus, L^μ_a and L^μ_b are unitarily equivalent. This proves assertion i).

ii) Choose $\mu = 0$ in the previous assertion.

iii) As mentioned above, by Lemma 4.3 we have (4.5). The same property holds for L^μ_b, that is, $\lambda^\mu_{b,k} \to \lambda^D_{b,k}$ as $\mu \to -\infty$. It follows from assertion (i) that $\lambda^D_{a,k} = \lambda^D_{b,k}$ for all $k \geq 1$ and have the same multiplicity. We conclude as above that L^D_a and L^D_b are unitarily equivalent.

Finally, another application of Theorem 4.2 shows that $\mathrm{Tr}(\mathrm{Ker}(\lambda - L^\mu_a)) = \mathrm{Tr}(\mathrm{Ker}(\lambda - L^\mu_b))$ for $\lambda \notin \sigma(L^D_a) = \sigma(L^D_b)$. □

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References

[1] W. Arendt, C.J.K. Batty, M. Hieber and F. Neubrander, *Vector-Valued Laplace Transforms and Cauchy Problems.* Second edition. Monographs in Mathematics, 96. Birkhäuser/Springer Basel AG, Basel, 2011.

[2] W. Arendt and R. Mazzeo, Spectral properties of the Dirichlet-to-Neumann operator on Lipschitz domains, *Ulmer Seminare* 2007.

[3] W. Arendt and R. Mazzeo, Friedlander's eigenvalue inequalities and the Dirichlet-to-Neumann semigroup, *Commun. Pure Appl. Anal.* 11 (2012), no. 6, 2201–2212.

[4] K. Astala and L. Päivärinta, Calderón's inverse conductivity problem in the plane, *Ann. of Math.* (2) 163 (2006), no. 1, 265–299.

[5] K. Astala, M. Lassas, and L. Päivärinta, Calderón's inverse problem for anisotropic conductivity in the plane, *Comm. Partial Differential Equations* 30 (2005), no. 1-3, 207–224.

[6] J. Behrndt and J. Rohleder, An inverse problem of Calderón type with partial data, *Comm. Partial Differential Equations* 37 (2012), no. 6, 1141–1159.

[7] P. Caro and K. Rogers, Global uniqueness for the Calderón problem with Lipschitz conductivities, http://arxiv.org/abs/1411.8001.

[8] D. Daners, Dirichlet problems on varying domains, *J. Diff. Eqs* 188 (2003) 591-624.

[9] E.B. Davies, *Heat Kernel and Spectral Theory*, Cambridge Tracts in Math. 92, Cambridge Univ. Press 1989.

[10] D. Dos Santos Ferreira, C.E. Kenig, M. Salo and G. Uhlmann, Limiting Carleman weights and anisotropic inverse problems, *Invent. Math.* 178 (2009), 119–171.

[11] A.F.M. ter Elst and E.M. Ouhabaz, Analysis of the heat kernel of the Dirichlet-to-Neumann operator, *J. Functional Analysis* 267 (2014) 4066-4109.

[12] A. Greenleaf, M. Lassas and G. Uhlmann, The Calderon problem for conormal potentials, I: Global uniqueness and reconstruction, *Comm. Pure Appl. Math.* 2003, 56, 328-352.

[13] B. Haberman and D. Tataru, Uniqueness in Calderón's problem with Lipschitz conductivities, *Duke Math. J.* 162 (2013), no. 3, 496–516.

[14] B. Haberman, Uniqueness in Calderón's problem for conductivities with unbounded gradient, http://arxiv.org/abs/1410.2201.

[15] V. Isakov, On uniqueness in the inverse conductivity problem with local data, *Inverse Probl. Imaging* 1 (2007), 95–105.

[16] O. Imanuvilov, G. Uhlmann and M. Yamamoto, The Calderón problem with partial data in two dimensions, *J. Amer. Math. Soc.* 23 (2010), 655–691.

[17] T. Kato, *Perturbation Theory for Linear Operators*. Second edition, Grundlehren der mathematischen Wissenschaften 132. Springer-Verlag, Berlin etc., 1980.

[18] C.E. Kenig, J. Sjöstrand and G. Uhlmann, The Calderón problem with partial data, *Ann. of Math.* 165 (2007), 567–591.

[19] C.E. Kenig and M. Salo, Recent progress in the Calderón problem with partial data, http://arxiv.org/abs/1302.4218.

[20] J. Lee and G. Uhlmann, Determining anisotropic real-analytic conductivities by boundary measurements, *Comm. Pure Appl. Math.* 42 (1989) 1097–1112.

[21] A. Nachman, Global uniqueness for a two-dimensional inverse boundary value problem, *Ann. of Math.* 1996, 143(2), 71–96.

[22] E.M. Ouhabaz, Invariance of closed convex sets and domination criteria for semigroups, *Potential Anal.* 5 (1996), no. 6, 611–625.

[23] E. M. Ouhabaz, *Analysis of Heat Equations on Domains.* London Math. Soc. Monographs, Princeton Univ. Press 2005.

[24] E.M. Ouhabaz, Gaussian upper bounds for heat kernels of second-order elliptic operators with complex coefficients on arbitrary domains, *J. Operator Theory* 51 (2004) 335-360.

[25] J. Rohleder, Strict inequality of Robin eigenvalues for elliptic differential operators on Lipschitz domains, *J. Math. Anal. Appl.* 418 (2014) 978–984.

[26] F. Schulz, On the unique continuation property of elliptic divergence form equations in the plane, *Math. Z.* 228 (1998), 201-206.

[27] J. Sylvester and G. Uhlmann, A global uniqueness theorem for an inverse boundary value problem, *Ann. of Math.* 1987, 125(1), 153–169.

[28] T. H. Wolff, Recent work on sharp estimates in second-order elliptic unique continuation problems, *J. Geom. Anal.* 3 (1993), no. 6, 621-650.

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